Solutions Exam QFT 23 January 2018

1a) $Z[J,J^\dagger] = \int \mathcal{D}\varphi \mathcal{D}\varphi^\dagger \, e^{i\int d^dx \{\mathcal{L} + J^\dagger \varphi + \varphi^\dagger J\}}$

1b) The Feynman rule in momentum space is given by the connected contribution to the 4-point correlation function $G^{(4)}$ to order λ.

$$G^{(4)}(x_1 \ldots x_4) = \frac{1}{Z[0,0,\lambda=0]} \int \mathcal{D}\varphi \mathcal{D}\varphi^\dagger \, e^{i\int d^dx \, \mathcal{L}_0} \Big\{ 1 - i\lambda \int d^dw \, (\varphi^\dagger\varphi)^2(w) \cdot$$
$$\cdot \, \varphi(x_1)\varphi^\dagger(x_2)\varphi(x_3)\varphi^\dagger(x_4) + O(\lambda^2) \Big\}$$

where $\mathcal{L}_0 = \partial_\mu \varphi^\dagger \partial^\mu \varphi - M^2 \varphi^\dagger \varphi$

The connected contribution at order λ is:

$$G_c^{(4)}(x_1 \ldots x_4)\Big|_{O(\lambda)} = -i\lambda \int d^dw \; 2 \cdot 2 \; iD(x_1 - w) \ldots \; iD(x_4 - w)$$

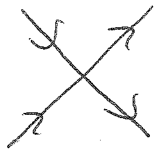
By replacing $iD(x_J - w) = \int \frac{d^dk_J}{(2\pi)^d} e^{-ik_J(x_J - w)} \frac{i}{k_J^2 - M^2 + i\varepsilon}$ we obtain:

$$G_c^{(4)}(x_1 \ldots x_4)\Big|_{O(\lambda)} = -4i\lambda \int d^dw \prod_{J=1}^{4} \int \frac{d^dk}{(2\pi)^d} e^{-ik_J(x_J - w)} \frac{i}{k_J^2 - M^2 + i\varepsilon}$$

$$= -4i\lambda \prod_{J=1}^{4} \int \frac{d^dk_J}{(2\pi)^d} e^{-ik_J x_J} \frac{i}{k_J^2 - M^2 + i\varepsilon} \cdot (2\pi)^d \delta^d\Big(\sum_{J=1}^{4} k_J\Big)$$

where $\int d^dw \, e^{i\sum_{J=1}^{4} k_J w} = (2\pi)^d \delta^d\Big(\sum_{J=1}^{4} k_J\Big)$

The Feynman rule is then obtained by "amputating" the external legs' factors $\prod_{J=1}^{4} \int \frac{d^dk_J}{(2\pi)^d} e^{-ik_J x_J} \frac{i}{k_J^2 - M^2 + i\varepsilon}$

thus providing  $-4i\lambda$

where the conservation of the total momentum flowing in the vertex $\left((2\pi)^d \delta^d\left(\sum_{J=1}^{4} k_J\right)\right)$ is understood.

Note that the arrows in the Feynman diagram indicate the flow of the particle-charge.

2a) The Euler-Lagrange equation is

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \varphi_a} = \frac{\delta \mathcal{L}}{\delta \varphi_a}$$

where in this case $\varphi_a = \varphi, \varphi^+$.

Taking the derivatives of $\mathcal{L}$ w.r.t. φ^+ we obtain the EoM for φ:

$$\partial_\mu D^\mu \varphi = -M^2 \varphi + ieA_\mu D^\mu \varphi$$

$$(\partial_\mu - ieA_\mu) D^\mu \varphi + M^2 \varphi = 0$$

Thus $$(D_\mu D^\mu + M^2)\varphi = 0$$

Analogously for the EoM for φ^+:

$$\partial_\mu D^\mu \varphi^+ = -M^2 \varphi^+ - ieA_\mu D^\mu \varphi^+$$

$$(\partial_\mu + ieA_\mu) D^\mu \varphi^+ + M^2 \varphi^+ = 0$$

Thus $$(D_\mu D^\mu + M^2)\varphi^+ = 0$$

2b) $$J^\mu = \frac{\delta \mathcal{L}}{\delta \partial_\mu \varphi} \delta\varphi + \frac{\delta \mathcal{L}}{\delta \partial_\mu \varphi^+} \delta\varphi^+$$

with $\delta\varphi = i\alpha\varphi$, $\delta\varphi^+ = -i\alpha\varphi^+$.

Thus $J^\mu = i\alpha\left(\varphi D^\mu \varphi^+ - \varphi^+ D^\mu \varphi\right)$

Extra (not required): Prove that $\partial_\mu J^\mu = 0$ using the EoM

3a) $D = d - [g]V - [\varphi]E_B - [\psi]E_F$

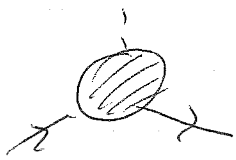
From dimensional analysis: $[\psi] = \frac{d-[\partial]}{2} = \frac{d-1}{2}$

$$[\varphi] = \frac{d-2[\partial]}{2} = \frac{d-2}{2}$$

$[g] = d - 2[\psi] - [\varphi] = \frac{4-d}{2}$

The theory is superrenormalizable for $[g] > 0$, which implies $d < 4$.

3b) For $d = 2$, $E_B = 1$, $E_F = 2$

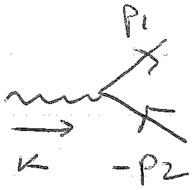


one has $D = 2 - V - 2\cdot\frac{1}{2} = 1 - V$

UV divergences can only be contained in loop corrections, and loop contributions to this amplitude ($E_B = 1$, $E_F = 2$) appear only for $V \geq 2$.

On the other hand, $D \geq 0$ — and thus the corresponding Feynman diagram is superficially UV divergent — only for $V \leq 1$, since $D = 1 - V$.

We conclude that the amplitude is UV finite, because all Feynman diagrams contributing to it are finite.

4a) 

$$\mathcal{A} = ig\,\bar u_s(\vec p_1)\,\gamma^\mu v_{s'}(\vec p_2)\,\varepsilon_{\mu z}(\vec k)$$

$$\mathcal{A}^\dagger = -ig\,\bar v_{s'}(\vec p_2)\,\gamma^\nu u_s(\vec p_1)\,\varepsilon_{\nu z}(\vec k)$$

$$X = \frac{g^2}{3}\sum_{ss'} \bar u_s(\vec p_1)\gamma^\mu v_{s'}(\vec p_2)\,\bar v_{s'}(\vec p_2)\gamma^\nu u_s(\vec p_1)\left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{M^2}\right)$$

$$= \frac{g^2}{3}\,\mathrm{Tr}\left[\frac{\not p_1 + m}{2m}\gamma^\mu \frac{\not p_2 - m}{2m}\gamma^\nu\right]\left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{M^2}\right)$$

$$= \frac{g^2}{12m^2}\,\mathrm{Tr}\left[\not p_1\gamma^\mu \not p_2\gamma^\nu - m^2\gamma^\mu\gamma^\nu\right]\left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{M^2}\right)$$

$$= \frac{g^2}{12m^2}\,\mathrm{Tr}\left[-\not p_1\gamma^\mu \not p_2\gamma_\mu + m^2\gamma^\mu\gamma_\mu + \frac{1}{M^2}\not p_1\not k\not p_2\not k - \frac{m^2}{M^2}\not k\not k\right]$$

Use $\gamma^\mu\not p_2\gamma_\mu = -2\not p_2$, $\not k\not p_2\not k = 2p_2\cdot k\,\not k - M^2\not p_2$ $(k^2 = M^2)$

$$X = \frac{g^2}{12m^2}\left(4p_1p_2 + \frac{8}{M^2}(p_1\cdot k)(p_2\cdot k) + 12m^2\right)$$

Rest Frame kinematics: $p_1 = (E,\vec p)$ $p_2 = (E,-\vec p)$ $k = (M,\vec 0)$

$p_1\cdot p_2 = \frac{M^2}{2} - m^2$ $\quad p_1\cdot k = p_2\cdot k = \frac{M^2}{2}$ $\quad E = \frac{M}{2}$

$$X = \frac{g^2}{3}\frac{M^2}{m^2}\left(1 + \frac{2m^2}{M^2}\right)$$

$$\Gamma_{\text{spin}1} = \frac{g^2}{12\pi}M\left(1 + \frac{2m^2}{M^2}\right)\sqrt{1 - \frac{4m^2}{M^2}}$$

4b) We only need to compare $X_{\text{spin}1}$ and $X_{\text{spin}0}$.
Requiring that $X_{\text{spin}0} \geq X_{\text{spin}1}$:

$$\frac{1}{2}\left(1 - \frac{4m^2}{M^2}\right) \geq \frac{1}{3}\left(1 + \frac{2m^2}{M^2}\right)$$

implies $\frac{m}{M} \leq \frac{1}{4}$. Thus the maximum is $\frac{m}{M} = \frac{1}{4}$.